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Problems of periodic solutions for a type of Duffing equation with state-dependent delay

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ABSTRACT

Sufficient criteria are established for the existence of periodic solutions to a type of Duffing equation with state-dependent delay, which improve and generalize some related results in the literature. The approach is based on Mawhin's continuation theorem. The significance of the present paper is that our results are relevant to delay by Lemma 2.1, which is different from the corresponding results of past work.

1. Introduction

The Duffing equation $x''(t) + g(x(t)) = p(t)$ has been studied by many researchers because of its widely applied background. In recent years, some important results have been obtained for the existence of periodic solutions to delay Duffing equations (see papers [1–4]). However, these results have been obtained in the case where τ is a constant or τ is only related to time t. We note that the resilient conditions, g is bounded and $\int_0^T p(t)\mathrm{d}t = 0$, were imposed on g and f in most of the papers.

In fact, delay is not only related to time t but is also related to the current state $x(t)$. As is well known, over the past several years it has become apparent that equations with state-dependent delays arise in several areas such as in population models [5], in models of cell productions [6], and in models of commodity price fluctuations [7]. Hence there is more practical significance in studying the Duffing equation with state-dependent delay.

In this paper, we will consider the following Duffing equation with state-dependent delay:

$$x''(t) + g(x(t - \tau(t, x(t)))) = p(t), \tag{1}$$

where $g \in C(\mathbb{R}, \mathbb{R})$; $p \in C(\mathbb{R}, \mathbb{R})$ with $p(t + T) = p(t)$; $\tau \in C(\mathbb{R}^2, \mathbb{R}_+)$ with $\tau(t + T, x) = \tau(t, x)$, $\forall x \in \mathbb{R}$; $T > 0$ is a given constant and $\mathbb{R}_+ = [0, \infty)$. By employing Mawhin's continuation theorem and some analysis approaches, a new result on the existence of periodic solutions is obtained on weaker conditions. Even if $\tau(t, x(t))$ degenerates into a constant τ or univariate function $\tau(t)$, here we extend the corresponding results of past work. It is worth stating that our results are related to the delay and the methods for estimating prior bounds are new, which are different from the corresponding results of past work [8–10].

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2. Preliminaries

In this section, we give some lemmas which will be used in this paper. Let

$$C_T = \{x | x \in C(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t),\ \forall t \in \mathbb{R}\}$$

with the norm

$$|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|, \quad \forall \varphi \in C_T$$

and

$$C_T^1 = \{x | x \in C^1(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t), \quad \forall t \in \mathbb{R}\}$$

with the norm

$$\|\varphi\| = \max_{t\in[0,T]} \{|\varphi|_0,\ |\varphi'|_0\}, \quad \forall \varphi \in C_T^1.$$

Clearly, C_T and C_T^1 are Banach spaces. Define a linear operator

$$L : D(L) \subset C_T^1 \to C_T, \qquad Lx = x'', \tag{2}$$

where $D(L) = \{x | x \in C^2(\mathbb{R}, \mathbb{R}),\ x(t+T) \equiv x(t)\}$, and a nonlinear operator

$$N : C_T^1 \to C_T, \qquad Nx = -g(x(t - \tau(t, x(t)))) + p(t). \tag{3}$$

It is easy to see

$$\operatorname{Ker} L = \{a,\ a \in \mathbb{R}\}, \qquad \operatorname{Im} L = \{y | y \in C_T, \int_0^T y(s)\mathrm{d}s = 0\}.$$

So $\operatorname{Im} L$ is closed in C_T and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$, then the operator L is a Fredholm operator with index zero. Define continuous projectors

$$P : C_T \to \operatorname{Ker} L, \quad Px = x(0)$$

and

$$Q : C_T \to C_T/\operatorname{Im} L, \quad Qy = \frac{1}{T}\int_0^T y(s)\mathrm{d}s.$$

It is easy to get

$$\operatorname{Im} P = \operatorname{Ker} L \quad \text{and} \quad \operatorname{Ker} Q = \operatorname{Im} L.$$

Set operators

$$L_P = L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L.$$

Then L_p has a unique continuous inverse operator L_P^{-1} on $\operatorname{Im} L$ defined by

$$(L_P^{-1}y)(t) = \int_0^T G(t, s)y(s)\mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} \dfrac{s(T-t)}{T}, & 0 \le s < t; \\ \dfrac{t(T-s)}{T}, & t \le s \le T. \end{cases} \tag{4}$$

Lemma 2.1. *Let $0 \le \alpha \le T$ be a constant, $s(t) \in C_T$ with $0 \le s(t) \le \alpha$. Then for $\forall x \in C_T^1$, we have*

$$\int_0^T |x(t - s(t)) - x(t)|^2 \mathrm{d}t \le \beta \int_0^T |x'(t)|^2 \mathrm{d}t,$$

where

$$\begin{aligned} \beta &= \max\left\{ \max_{\sigma\in[-\alpha,0]} \int_0^{\sigma+\alpha} s(t)\mathrm{d}t,\ \max_{\sigma\in[T-\alpha,T]} \int_\sigma^T s(t)\mathrm{d}t,\ \max_{\sigma\in[0,T-\alpha]} \int_\sigma^{\sigma+\alpha} s(t)\mathrm{d}t \right\} \\ &= \max_{\sigma\in[0,T-\alpha]} \int_\sigma^{\sigma+\alpha} s(t)\mathrm{d}t. \end{aligned}$$

Proof. Since

$$\begin{aligned}
\int_0^T |x(t-s(t))-x(t)|^2\mathrm{d}t &= \int_0^T \left|\int_{t-s(t)}^t x'(\sigma)\mathrm{d}\sigma\right|^2 \mathrm{d}t \\
&\le \int_0^T s(t)\int_{t-s(t)}^t |x'(\sigma)|^2\mathrm{d}\sigma\,\mathrm{d}t \\
&\le \int_0^T \int_{t-\alpha}^t s(t)|x'(\sigma)|^2\mathrm{d}\sigma\,\mathrm{d}t \\
&= \int_{-\alpha}^0\int_0^{\sigma+\alpha} s(t)|x'(\sigma)|^2\mathrm{d}t\mathrm{d}\sigma + \int_0^{T-\alpha}\int_\sigma^{\sigma+\alpha} s(t)|x'(\sigma)|^2\mathrm{d}t\mathrm{d}\sigma + \int_{T-\alpha}^T\int_\sigma^T s(t)|x'(\sigma)|^2\mathrm{d}t\mathrm{d}\sigma \\
&\le \max_{\sigma\in[-\alpha,0]}\left\{\int_0^{\sigma+\alpha} s(t)\mathrm{d}t\right\}\int_{-\alpha}^0 |x'(\sigma)|^2\mathrm{d}\sigma + \max_{\sigma\in[0,T-\alpha]}\left\{\int_\sigma^{\sigma+\alpha} s(t)\mathrm{d}t\right\}\int_0^{T-\alpha}|x'(\sigma)|^2\mathrm{d}\sigma \\
&\quad + \max_{\sigma\in[T-\alpha,T]}\left\{\int_\sigma^T s(t)\mathrm{d}t\right\}\int_{T-\alpha}^T |x'(\sigma)|^2\mathrm{d}\sigma \\
&\le \beta\int_0^T |x'(\sigma)|^2\mathrm{d}\sigma.
\end{aligned}$$

Hence we obtain that Lemma 2.1 holds. □

Now, let $\tau(t)\ge 0\in C_T$, then there must exist two integers k and m with $k\ge 0$ and $m\ge 1$ such that

$$\tau(t)\in[kT,(k+m)T],\qquad \tau(t)\notin(0,kT)\cup((k+m)T,+\infty). \tag{5}$$

Denote

$$E_i=\{t: t\in[0,T],\ \tau(t)\in[(k+i)T,(k+i+1)T]\},\quad (i=0,1,2,\dots,m-1),$$
$$\tau_0(t)=\begin{cases}\tau(t), & t\in E_0,\\ (k+1)T, & t\in[0,T]\setminus E_0\end{cases}$$

and

$$\tau_j(t)=\begin{cases}\tau(t), & t\in E_j,\\ (k+j)T, & t\in[0,T]\setminus E_j.\end{cases}$$

Obviously, $\bigcup_{i=0}^{m-1}E_i=[0,T]$; $(k+1)T-\tau_0(t)\in[0,T]$ and $\tau_j(t)-(k+j)T\in[0,T]$, $\forall t\in[0,T]$, $(j=1,2,\dots,m-1)$. Let

$$\delta_0=\sup_{t\in[0,T]}[(k+1)T-\tau_0(t)],\qquad \delta_j=\sup_{t\in[0,T]}[\tau_j(t)-(k+j)T],$$

then we have $\delta_0,\ \delta_{m-1}\in[0,T]$, $\delta_j=T$, $(j=1,2,\dots,m-2)$.

Lemma 2.2. *Let $\tau(t,x(t))\in C_T$ satisfying (5) and $x\in C_T^1$, then*

$$\int_0^T |x(t-\tau(t,x(t)))-x(t)|^2\mathrm{d}t \le \left(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\right)\int_0^T |x'(t)|^2\mathrm{d}t,\quad \textit{for } 2<m<\infty,$$
$$\int_0^T |x(t-\tau(t,x(t)))-x(t)|^2\mathrm{d}t \le (\beta_0+\beta_{m-1})\int_0^T |x'(t)|^2\mathrm{d}t,\quad \textit{for } 1\le m\le 2,$$

where

$$\beta_0=\max_{\sigma\in[0,T-\delta_0]}\int_\sigma^{\sigma+\delta_0}\tau(t,x(t))\mathrm{d}t,$$
$$\beta_{m-1}=\max_{\sigma\in[0,T-\delta_{m-1}]}\int_\sigma^{\sigma+\delta_{m-1}}\tau(t,x(t))\mathrm{d}t,$$
$$\beta_j=\int_0^T\tau(t,x(t))\mathrm{d}t,\quad j=1,2,\dots,m-2.$$

Proof. We show that Lemma 2.2 holds in case of $2 < m < \infty$. From

$$\begin{aligned}\int_0^T |x(t-\tau(t,x(t)))-x(t)|^2\mathrm{d}t &= \sum_{i=0}^{m-1}\int_{E_i}|x(t)-x(t-\tau_i(t,x(t)))|^2\mathrm{d}t\\ &= \int_{E_0}|x(t)-x(t-\tau_0(t,x(t)))|^2\mathrm{d}t+\int_{E_{m-1}}|x(t)-x(t-\tau_{m-1}(t,x(t)))|^2\mathrm{d}t\\ &\quad+\sum_{i=1}^{m-2}\int_{E_i}|x(t)-x(t-\tau_i(t,x(t)))|^2\mathrm{d}t\\ &= \int_{E_0}|x(t)-x(t-\tau_0(t,x(t))+(k+1)T)|^2\mathrm{d}t\\ &\quad+\int_{E_{m-1}}|x(t)-x(t-\tau_{m-1}(t,x(t))+(k+m-1)T)|^2\mathrm{d}t\\ &\quad+\sum_{i=1}^{m-2}\int_{E_i}|x(t)-x(t-\tau_i(t,x(t))+(k+i)T)|^2\mathrm{d}t,\end{aligned}$$

by Lemma 2.1, we get

$$\int_0^T |x(t-\tau(t,x(t)))-x(t)|^2\mathrm{d}t \le \left(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\right)\int_0^T|x'(t)|^2\mathrm{d}t.$$

In case of $1 \le m \le 2$, the result is clear. □

Lemma 2.3 ([11]). *Let $x \in C_T^1$, and there exists a constant $\xi \in \mathbb{R}$ such that $x(\xi)=0$. Then we have*

$$\int_0^T |x(t)|^2\mathrm{d}t \le \frac{T^2}{\pi^2}\int_0^T |x'(t)|^2\mathrm{d}t.$$

Lemma 2.4 ([12]). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(1) $Lx \ne \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\forall \lambda \in (0,1)$,
(2) $Nx \notin Im\,L$, $\forall x \in \partial\Omega \cap Ker\ L$,
(3) $deg\{QN, \Omega \cap Ker\ L, 0\} \ne 0$.

Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap D(L)$.

3. Existence of a periodic solution for Eq. (1)

Theorem 3.1. *Suppose that there exist constants $d > 0$ and $L \ge 0$ such that*

(H1) $xg(x) < 0$ *and* $|g(x)| > |p|_0$, *whenever* $|x| > d$,
(H2) $|g(x_1)-g(x_2)| \le L|x_1-x_2|$, $\forall x_1, x_2 \in \mathbb{R}$.

Then Eq. (1) has at least one T-periodic solution, if

$$\frac{LT}{\pi}\left(\beta_0+\beta_{m-1}+\sum_{i=1}^{m-2}\beta_j\right)^{\frac{1}{2}} < 1.$$

Proof. Consider the operator equation:

$$Lx = \lambda Nx, \tag{6}$$

where L and N are defined by (2) and (3) respectively. Let $x(t)$ be an arbitrary T-periodic solution of Eq. (6). Suppose t_0 and t_1 be the global maximum point and global minimum point of $x(t)$ on $[0,T]$ respectively. Then $x'(t_0)=0$, $x''(t_0) \le 0$, and

$$g(x(t_0-\tau(t_0,x(t_0)))) \ge p(t_0) \ge -|p|_0. \tag{7}$$

Similarly, we can obtain

$$g(x(t_1-\tau(t_1,x(t_1)))) \le p(t_1) \le |p|_0. \tag{8}$$

Case 1. If $g(x(t_0-\tau(t_0,x(t_0)))) > |p|_0$, from (8) we get there exists a point $\eta \in [0,T]$ such that $g(x(\eta-\tau(\eta,x(\eta)))) = |p|_0$. So by assumption (H1) we have

$$|x(\eta-\tau(\eta,x(\eta)))| \le d.$$

Case 2. If $g(x(t_0-\tau(t_0,x(t_0)))) \le |p|_0$, from (7) and assumption (H1) we get

$$|x(t_0-\tau(t_0,x(t_0)))| \le d.$$

So in either case (1) or case (2) we always obtain that there is a point $\xi \in [0,T]$ such that

$$|x(\xi-\tau(\xi,x(\xi)))| \le d.$$

Since $x(t)$ is a T-periodic function, then there is an integer k and a constant $t^* \in [0,T]$ such that $\xi-\tau(\xi,x(\xi)) = kT+t^*$. Then we get

$$|x|_0 \le |x(t^*)| + \int_0^T |x'(s)|\mathrm{d}s \le d + \int_0^T |x'(s)|\mathrm{d}s. \tag{9}$$

In the following proof, let $\triangle_1 = \{t \in [0,T] : |x(t)| > d\}$, $\triangle_2 = \{t \in [0,T] : |x(t)| \le d\}$, $\|u\|_2 = \left(\int_0^T |u(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}$. Multiplying both sides of (6) by $x(t)$ and integrating them on $[0,T]$, from Lemma 2.2 we get

$$-\int_0^T (x'(t))^2\mathrm{d}t = -\lambda\int_0^T g(x(t-\tau(t,x(t))))x(t)\mathrm{d}t + \lambda\int_0^T p(t)x(t)\mathrm{d}t,$$

i.e.,

$$\begin{aligned}
\int_0^T |x'(t)|^2\mathrm{d}t &= \lambda\int_0^T g(x(t-\tau(t,x(t))))x(t)\mathrm{d}t - \lambda\int_0^T p(t)x(t)\mathrm{d}t\\
&= \lambda\int_0^T (g(x(t-\tau(t,x(t)))) - g(x(t)))x(t)\mathrm{d}t + \lambda\int_0^T g(x(t))x(t)\mathrm{d}t - \lambda\int_0^T p(t)x(t)\mathrm{d}t\\
&= \lambda\int_0^T (g(x(t-\tau(t,x(t)))) - g(x(t)))x(t)\mathrm{d}t + \lambda\int_{\triangle_1} g(x(t))x(t)\mathrm{d}t + \lambda\int_{\triangle_2} g(x(t))x(t)\mathrm{d}t\\
&\quad -\lambda\int_0^T p(t)x(t)\mathrm{d}t\\
&\le \int_0^T |g(x(t-\tau(t,x(t)))) - g(x(t))||x(t)|\mathrm{d}t + g_d\int_0^T |x(t)|\mathrm{d}t + \int_0^T |p(t)||x(t)|\mathrm{d}t\\
&\le L\int_0^T |x(t-\tau(t,x(t))) - x(t)||x(t)|\mathrm{d}t + g_d T^{\frac{1}{2}}\|x\|_2 + \|p\|_2\|x\|_2\\
&\le L\left(\int_0^T |x(t-\tau(t,x(t))) - x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\|x\|_2 + g_d T^{\frac{1}{2}}\|x\|_2 + \|p\|_2\|x\|_2\\
&\le L\left(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}\|x\|_2 + g_d T^{\frac{1}{2}}\|x\|_2 + \|p\|_2\|x\|_2,
\end{aligned} \tag{10}$$

here $g_d = \max_{t\in\triangle_2}|g(x(t))|$. Now we denote $u(t) = x(t) - x(t^*)$, where t^* is defined by (9). Then we have

$$\begin{aligned}
|x(t)| &\le |x(t^*)| + |x(t) - x(t^*)|\\
&\le d + |u(t)|.
\end{aligned}$$

From the well known Minkowski inequality, we get

$$\left(\int_0^T |x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \le T^{\frac{1}{2}}d + \left(\int_0^T |u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}. \tag{11}$$

Since $u(t^*) = 0$, $u(t+T) = u(t)$, $u'(t) = x'(t)$, by Lemma 2.3 we have

$$\left(\int_0^T |u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \le \frac{T}{\pi}\left(\int_0^T |u'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} = \frac{T}{\pi}\left(\int_0^T |x'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

combining with (11), we have

$$\left(\int_0^T |x(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \le T^{\frac{1}{2}} d + \frac{T}{\pi}\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \tag{12}$$

It follows from (10) and (12) that

$$\begin{aligned}
\int_0^T |x'(t)|^2 \mathrm{d}t \le\ & L\left(\beta_0 + \beta_{m-1} + \sum_{i=1}^{m-2} \beta_j\right)^{\frac{1}{2}} \left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \left(T^{\frac{1}{2}} d + \frac{T}{\pi}\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right) \\
& + (g_d T^{\frac{1}{2}} + \|p\|_2)\left(T^{\frac{1}{2}} d + \frac{T}{\pi}\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right) \\
=\ & \frac{LT}{\pi}\left(\beta_0 + \beta_{m-1} + \sum_{i=1}^{m-2} \beta_j\right)^{\frac{1}{2}} \int_0^T |x'(t)|^2 \mathrm{d}t \\
& + L\left(\beta_0 + \beta_{m-1} + \sum_{i=1}^{m-2} \beta_j\right)^{\frac{1}{2}} T^{\frac{1}{2}} d \left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
& + (g_d T^{\frac{1}{2}} + \|p\|_2)\frac{T}{\pi}\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} + (g_d T^{\frac{1}{2}} + \|p\|_2) T^{\frac{1}{2}} d.
\end{aligned} \tag{13}$$

In view of $\frac{LT}{\pi}\left(\beta_0 + \beta_{m-1} + \sum_{i=1}^{m-2} \beta_j\right)^{\frac{1}{2}} < 1$, from (13) we obtain that there exists a constant $M_1 > 0$ such that

$$\int_0^T |x'(t)|^2 \mathrm{d}t \le M_1.$$

So from (9), we have

$$|x|_0 \le d + T^{\frac{1}{2}} M_1^{\frac{1}{2}} := M_2.$$

By (6) it is easy to see that

$$|x''(t)| \le g_{M_2} + |p|_0 := M_3, \tag{14}$$

here $g_{M_2} = \max_{|x| \le M_2} |g(x)|$. As $x(0) = x(T)$, it follows that there is a constant $\eta \in [0, T]$ such that $x'(\eta) = 0$. So by (14) we have

$$|x'|_0 \le |x'(\eta)| + \int_0^T |x''(s)| \mathrm{d}s \le TM_3 := M_4.$$

Obviously, M_2 and M_4 are independent of λ and x. Take $\Omega = \{x | x \in X, |x|_0 < M_2, |x'|_0 < M_4\}$, $\Omega_1 = \{x | x \in \operatorname{Ker} L, Nx \in \operatorname{Im} L\}$, clearly $\forall x \in \Omega_1$, $x \equiv c$ is a constant and $g(c) = p(t)$, by assumption (H1) we have $|c| \le d$, hence $\Omega_1 \subset \Omega$ and the conditions (1) and (2) of Lemma 2.4 are satisfied. Now we show that (3) of Lemma 2.4 is true. Let

$$H(x, \mu) = \mu x - \frac{1-\mu}{T}\int_0^T (g(x) - p(t)) \mathrm{d}t.$$

It is easy to see that

$$H(x, \mu) \ne 0, \quad \forall (x, \mu) \in (\partial\Omega \cap \operatorname{Ker} L) \times [0, 1].$$

Hence

$$\begin{aligned}
deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} &= deg\{H(x, 0), \Omega \cap \operatorname{Ker} L, 0\} \\
&= deg\{H(x, 1), \Omega \cap \operatorname{Ker} L, 0\} \\
&= deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \ne 0.
\end{aligned}$$

Meanwhile, from (3) and (4), we know that N is L—-compact on $\bar{\Omega}$. So by applying Lemma 2.4, we obtain that Eq. (1) has at least a T-periodic solution in $\bar{\Omega}$.

4. Applications

There are extensively applied backgrounds in oscillating theory for Eq. (1). As applications, we here give the following example:

$$x''(t) - \frac{x^3\left(t - \frac{1}{40}\tau\right)}{1 + x^2\left(t - \frac{1}{20}\tau\right)} = \sin t, \tag{15}$$

where $g(x) = -\frac{x^3}{1+x^2}$, $\tau = \tau(t, x(t)) = \frac{1}{80}|\cos(t + 10x(t))|$. Clearly, $L = 1$, $T = 2\pi$, $\tau(t, x) \in [0, 4\pi]$, $\forall t \in [0, 2\pi]$. So $k = 0$, $m = 2$, $\delta_0 = \delta_1 = 2\pi$, $\beta_0 = \beta_1 = \frac{\pi}{40}$. So $\frac{LT}{\pi}(\beta_0 + \beta_{m-1})^{\frac{1}{2}} < 1$. Applying Theorem 3.1, Eq. (15) has at least one 2π-periodic solution.

References

[1] S. Ma, Z. Wang, J. Yu, Coincidence degree and periodic solutions of Duffing equations, Nonlinear Anal. 34 (1998) 443–460.
[2] D. Hao, S. Ma, Semilinear Duffing equations crossing resonance points, J. Differential Equations 133 (1997) 98–116.
[3] B. Liu, Boundedness of solutions for semilinear Duffing equations, J. Differential Equations 145 (1998) 119–144.
[4] B. Liu, Boundedness in nonlinear oscillations at resonance, J. Differential Equations 153 (1999) 142–174.
[5] J. Bélair, Population Models with State-Dependent Delays, in: Lecture Notes in Pure Appl. Math., Dekker, New York, 1991.
[6] M.C. Mackey, J. Milton, Feedback delays and the origin of blood cell dynamics, Comm. Theor. Biol. 1 (1990) 229–327.
[7] M.C. Mackey, Commodity price fluctuations: Price dependent delays and continearities as explanatory factors, J. Econom. Theory 48 (1989) 487–509.
[8] S. Lu, W. Ge, Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument, Nonlinear Anal. 56 (2004) 501–514.
[9] S. Lu, W. Ge, Z. Zheng, Periodic solutions for a kind of Rayleigh equation with a deviating argument, Appl. Math. Lett. 17 (2004) 443–449.
[10] G. Wang, A priori bounds for periodic solutions of a delay Rayleigh equation, Appl. Math. Lett. 12 (1999) 41–44.
[11] X. Liu, M. Jia, R. Ren, On the existence and uniqueness of periodic solutions to a type of Duffing equation with complex deviating argument, Acta. Math. Sci. 27 (2007) 037–044. in Chinese.
[12] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, Berlin, 1977.